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A SINGULAR SINGULARLY-PERTURBED LINEAR BOUNDARY VALUE PROBLEM

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Abstract

We consider the asymptotic solution of boundary value problems for the vector system

$$\begin{cases} \dot{x} = A(t,\varepsilon)x + B(t,\varepsilon)y + C(t,\varepsilon) \\ \varepsilon\dot{y} = E(t,\varepsilon)x + F(t,\varepsilon)y + G(t,\varepsilon) \end{cases}$$

as $\varepsilon \to 0$ under the assumption that the matrix $F(t,0)$ is singular. A full set of asymptotic solutions is constructed assuming that $F(t,0)$ can be block-diagonalized, the reduced problem is consistent, and a new stability condition holds. Boundary value problems are then solvable if an appropriate "boundary" matrix is nonsingular for $\varepsilon \neq 0$. Such problems arise in optimal control theory, among other applications.

(See form 1473)

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A Singular Singularly-Perturbed Linear Boundary Value Problem

1. Introduction.

Let us consider a linear system of the form

$$(1)\qquad \begin{cases} \dot{x} = A(t,\varepsilon)x + B(t,\varepsilon)y + C(t,\varepsilon) \\ \\ \varepsilon\dot{y} = E(t,\varepsilon)x + F(t,\varepsilon)y + G(t,\varepsilon) \end{cases}$$

for vectors x and y of dimensions n and m, respectively, for a small positive parameter ε, and for a finite t interval, say $0 \leq t \leq 1$. It is natural to consider (1) subject to a list of $n + m$ linearly independent boundary conditions of the form

$$(2)\qquad \sum_{j=0}^{1} (R_j(\varepsilon)x(j) + S_j(\varepsilon)y(j)) = c(\varepsilon)$$

and study the asymptotic solution of (1) - (2) as $\varepsilon \to 0$.

We recall that rather classical methods can be used to asymptotically solve the "regular" singularly perturbed problem (1) - (2) when $F(t,0)$ satisfies an exponential dichotomy, i.e. its eigenvalues have either a positive or a negative real part throughout $0 \leq t \leq 1$ (cf., e.g., O'Malley (1969a), Harris (1973), or Ferguson (1975)). When $F(t,0)$ is everywhere stable, for example, they show that the initial value problem for (1) has a unique solution which converges as $\varepsilon \to 0$ for $t > 0$ to the solution of the reduced system

$$(3)\quad \begin{cases} \dot{X}_0 = A(t,0)X_0 + B(t,0)Y_0 + C(t,0) \\ 0 = E(t,0)X_0 + F(t,0)Y_0 + G(t,0) \end{cases}$$

subject to the initial condition $X_0(0) = x(0)$. For the analogous terminal value problem, however, the solution would then be exponentially large as $\varepsilon \to 0$ for $t < 1$. More generally, boundary layers (regions of nonuniform convergence) of thickness $O(\varepsilon)$ must be expected at each endpoint for regular problems when the limiting solution within $(0,1)$ is bounded, and this limiting solution must satisfy the reduced system (3) and n boundary conditions determined by an appropriate combination of the original conditions (2) evaluated at $\varepsilon = 0$. Because such limiting solutions involve only n boundary conditions, m linearly independent solutions of the homogeneous form of (1) are of boundary layer type, i.e. they are asymptotically negligible away from the endpoints. Considerable progress, then, has been made in determining which regular singularly-perturbed linear boundary value problems have limiting solutions as $\varepsilon \to 0$, what boundary value problems these limits satisfy within $(0,1)$, and the nature of the endpoint boundary layers. The results are, however, more complicated than for scalar problems (cf. O'Malley (1969b) and O'Malley and Keller (1968)). In addition to its direct utility, that information is useful in analyzing nonlinear problems (cf. Hoppensteadt (1971)) and in designing numerical algorithms (cf. Flaherty and O'Malley (1977)).

Here, we shall consider "singular" problems where $F(t,0)$ is singular and of constant rank throughout $0 \le t \le 1$. Their analysis and the behavior of their solutions are considerably more complicated than for regular problems. Specifically, we shall find that the asymptotic analysis of singular

problems involves a consistency condition which did not occur for regular problems, a new stability requirement, and the occurrence of other (thicker) boundary layer regions of nonuniform convergence. These singular problems are less complicated, however, than turning point problems where $F(t,0)$ is singular at isolated points (cf. Levinson (1951), Wasow (1965), and Olver (1977)). Fundamental matrices for homogeneous systems (1) without turning points can be constructed as in Turrittin (1952), and they could, in theory, be used to asymptotically solve nonhomogeneous problems via variation of parameters.

Our interest in such problems arose in analyzing nearly singular optimal control problems (cf. O'Malley and Jameson (1975, 1977)) and in devising methods for the numerical integration of stiff differential equations (cf. Flaherty and O'Malley (1977)). The technique we use generalizes that developed for singular arc computations (cf. Goh (1966) and Robbins (1967)). Closely related methods are given in O'Malley and Flaherty (1977) and O'Malley (1978) for certain nonlinear problems.

The simplest control example is given by

$$\begin{cases} \dot{x} = -y, & x(1) = 0 \\ \varepsilon\dot{y} = -x, & y(0) = -1. \end{cases}$$

Here, $-x/\varepsilon$ represents an optimal control and y, the corresponding state of a nearly singular control problem (cf. O'Malley and Jameson (1975)). The solution is

$$x(t) = -\sqrt{\varepsilon}\ (e^{-t/\sqrt{\varepsilon}} - e^{-1/\sqrt{\varepsilon}}\ e^{-(1-t)/\sqrt{\varepsilon}})/(1 + e^{-2/\sqrt{\varepsilon}})$$

and $y = -\dot{x}$, so the limiting solution

$$(x(t),y(t)) \sim -(\sqrt{\varepsilon},1)e^{-t/\sqrt{\varepsilon}}$$

is asymptotically trivial for $t > 0$ and features a $O(\sqrt{\varepsilon})$ boundary layer at $t = 0$. Note, in particular, that the corresponding control acts like an initial delta-function impulse.

2. A Transformed Problem and the Corresponding Reduced System.

Under rather mild assumptions, the singular matrix $F(t,0)$ can be block-diagonalized (cf. Sibuya (1958, 1966) and Chapter VII of Wasow (1965)). We shall simply assume

(H1) *that there exists a smooth nonsingular matrix* $P(t)$ *such that*

$$P^{-1}(t)F(t,0)P(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & F_5(t,0) & 0 \\ 0 & 0 & F_9(t,0) \end{bmatrix}$$

where $-F_5$ *and* F_9 *are stable matrices* (i.e., their eigenvalues have negative real parts) *throughout* $0 \le t \le 1$ *of dimensions* $m_2 \times m_2$ *and* $m_3 \times m_3$, *respectively, with* $m = m_1 + m_2 + m_3$, $m_1 > 0$.

Hypotheses guaranteeing the existence of P are given in Wasow (1964) and elsewhere. (We note that an analogous trichotomy was used by Hoppensteadt

and Miranker (1976), except that they allowed $F(t,0)$ to have purely imaginary, but nonzero, eigenvalues.) In analogy with the parallel situation in singular optimal control (cf. Jacobson (1971) or Anderson (1973)), we might call problems where $m = m_1$ totally singular and those where $m > m_1$ partially singular. (If either $m_2 = 0$ or $m_3 = 0$, it may be more convenient to use a singular value decomposition $F = UDV$ where U and V are orthogonal and D is diagonal with the eigenvalues of $\sqrt{F'F}$. One then uses the transformed variable $z = U'y$ (cf. O'Malley (1978)).

In general, we introduce

$$P^{-1}y = (y_1' \; y_2' \; y_3')' \tag{5}$$

in (1) and obtain the equivalent system

$$\left\{\begin{aligned} \dot{x} &= A(t,\varepsilon)x + B_1(t,\varepsilon)y_1 + B_2(t,\varepsilon)y_2 + B_3(t,\varepsilon)y_3 + C(t,\varepsilon) \\ \varepsilon\dot{y}_1 &= E_1(t,\varepsilon)x + \varepsilon F_1(t,\varepsilon)y_1 + \varepsilon F_2(t,\varepsilon)y_2 + \varepsilon F_3(t,\varepsilon)y_3 + G_1(t,\varepsilon) \\ \varepsilon\dot{y}_2 &= E_2(t,\varepsilon)x + \varepsilon F_4(t,\varepsilon)y_1 + F_5(t,\varepsilon)y_2 + \varepsilon F_6(t,\varepsilon)y_3 + G_2(t,\varepsilon) \\ \varepsilon\dot{y}_3 &= E_3(t,\varepsilon)x + \varepsilon F_7(t,\varepsilon)y_1 + \varepsilon F_8(t,\varepsilon)y_2 + F_9(t,\varepsilon)y_3 + G_3(t,\varepsilon) \end{aligned}\right. \tag{6}$$

where, in blocks compatible with (4),

$$P^{-1}(FP - \varepsilon\dot{P}) = \begin{bmatrix} \varepsilon F_1 & \varepsilon F_2 & \varepsilon F_3 \\ \varepsilon F_4 & F_5 & \varepsilon F_6 \\ \varepsilon F_7 & \varepsilon F_8 & F_9 \end{bmatrix},$$

$$BP = [B_1 \ B_2 \ B_3], \quad P^{-1}E = [E_1' \ E_2' \ E_3']', \quad \text{and} \quad P^{-1}G = [G_1' \ G_2' \ G_3']'.$$

Experience with singular perturbation problems leads us to expect that the limiting solution to (6) within $(0,1)$ will satisfy the reduced system obtained by setting $\varepsilon = 0$ in (6), i.e.

$$(7) \quad \begin{cases} \dot{X}_0 = A_0X_0 + B_{10}Y_{10} + B_{20}Y_{20} + B_{30}Y_{30} + C_0 \\ 0 = E_{10}X_0 + G_{10} \\ 0 = E_{20}X_0 + F_{50}Y_{20} + G_{20} \\ 0 = E_{30}X_0 + F_{90}Y_{30} + G_{30} \end{cases}$$

where, e.g., $A_0 = A(t,0)$. For any solution of (7),

$$(8) \quad \begin{cases} Y_{20} = -F_{50}^{-1}(E_{20}X_0 + G_{20}), \\ Y_{30} = -F_{90}^{-1}(E_{30}X_0 + G_{30}) \end{cases}$$

and there remains the m_1 linear equations

$$(9) \quad E_{10}X_0 = -G_{10}$$

and the n^{th} order system of differential equations

$$(10) \quad \dot{X}_0 = H_0X_0 + B_{10}Y_{10} + J_0$$

to determine the m_1-vector Y_{10} and the n-vector X_0. Here $H_0 = A_0 - B_{20}F_{50}^{-1}E_{20} - B_{30}F_{90}^{-1}E_{30}$ and $J_0 = C_0 - B_{20}F_{50}^{-1}G_{20} - B_{30}F_{90}^{-1}G_{30}$.

In order to solve (7), then, we'll assume

(H2) G_{10} <u>is in the range of</u> E_{10} <u>for</u> $0 \le t \le 1$

and

(H3) $-E_{10}B_{10}$ <u>is stable throughout</u> $0 \le t \le 1$.

Differentiating (9) we have

$$E_{10}\dot{X}_{10} = -\dot{E}_{10}X_{10} - \dot{G}_{10},$$

so (10) and (H3) allow us to solve for Y_{10} as

(11) $$Y_{10} = (-E_{10}B_{10})^{-1}[(E_{10}H_0 + \dot{E}_{10})X_{10} + (E_{10}J_0 + \dot{G}_{10})],$$

and there remains a nonhomogeneous system for X_0. To determine X_0, it is convenient to introduce the projection

(12) $$E = I_n - QE_{10}$$

where

(13) $$Q = B_{10}(E_{10}B_{10})^{-1}.$$

We note that $E_{10}E = 0$ while $E_{10}Q = I_{m_1}$ and $EB_{10} = EQ = 0$. Moreover,

E_{10} and B_{10} have rank m_1 since $E_{10}B_{10}$ has (full) rank m_1. Thus, $\mathcal{E}$ has rank $n - m_1 \geq 0$. Indeed, Q is nearly a generalized inverse of E_{10} (cf. Campbell, Meyer, and Rose (1976)). Using (9), (12) implies that

$$X_0 = \mathcal{E}X_0 - QG_{10} \tag{14}$$

and (10), (11), and (14) imply the linear system

$$(\mathcal{E}X_0)^{\bullet} = \mathcal{K}(\mathcal{E}X_0) + \mathcal{L} \tag{15}$$

for $\mathcal{E}X_0$ where $\mathcal{K} = \mathcal{E}H_0 - Q\dot{E}_{10}$ and $\mathcal{L} = -\mathcal{K}QG_{10} + \mathcal{E}J_0 + \dot{Q}G_{10}$. Under hypotheses (H1) - (H3), then, the solution X_0 of the reduced system (7) will be completely and uniquely determined up to later specification of a boundary value for $\mathcal{E}X_0$. It is perhaps most natural to use the condition

$$\mathcal{E}(j)X_0(j) = \mathcal{E}(j)x(j), \quad j = 0 \text{ or } 1, \tag{16}$$

presuming a boundary value (16) is supplied by (2). Other possibilities should also be considered, however.

Note that our manipulations allowed us to determine $E_{10}X_0$ from the linear equation (9) and the remaining "component" $\mathcal{E}X_0$ of X_0 from an end value problem like (15) - (16). The alternative problem character of the solution for X_0 (cf. Cesari (1975)) makes it quite different from the more straightforward solution of regular problems.

If (H2) fails, the reduced system (7) is inconsistent, but irrelevant (cf. O'Malley (1978)). A simplified example is provided by $\varepsilon\dot{y} = 1$, $y(0) = 0$ where the limiting solution for $t > 0$ is unbounded like t/ε. For regular problems, the reduced problem (3) is necessarily consistent,

but inconsistency of (3) would occur here if (H2) failed. Although we have only used the nonsingularity of $E_{10}B_{10}$ to define the reduced problem, the stability assumption (H2) is generally needed in order for a limiting solution to exist. A simple example is provided by $\dot{x} = -y$, $\varepsilon\dot{y} = x$, $x(1) = 0$, $y(0) = 1$ which has the solution $(x,y) = (-\sqrt{\varepsilon}\sin t/\sqrt{\varepsilon}, \cos t/\sqrt{\varepsilon})$ for which there is no limit as $\varepsilon \to 0$. We note that by changing the sign so that $\varepsilon\dot{y} = -x$, $E_{10}B_{10} < 0$, and we have a limiting solution. No such stability assumption was required for regular problems where $m_1 = 0$. If $E_{10}B_{10} = 0$, further differentiation of (9) might allow one to determine Y_{10}, just as singular arcs of higher order are obtained in control (cf., e.g., Robbins (1967)). The structure of the asymptotic solutions will then differ considerably from when (H3) holds. An example, arising in optimal control, is

$$\begin{cases} \dot{x}_1 = -y_1 + x_2, & x_1(1) = 0 \\ \dot{x}_2 = -x_1, & x_2(1) = 0 \\ \varepsilon\dot{y}_1 = \varepsilon y_2, & y_1(0) = 1 \\ \varepsilon\dot{y}_2 = -x_2 - \varepsilon y_1, & y_2(0) = 0 \end{cases}$$

(cf. O'Malley and Jameson (1977)). Here, the asymptotic solution is given by $x_1 = -\dot{x}_2$ and $y_2 = \dot{y}_1$ where $x_2 = 2\sqrt{\varepsilon}\,\mathrm{Im}\,[ce^{-\omega t/\sqrt[4]{\varepsilon}}]$ and $y_1 = 2\,\mathrm{Re}\,[ce^{-\omega t/\sqrt[4]{\varepsilon}}]$ for $c = (1 - i\sqrt{(1 + i\sqrt{\varepsilon})/(1 - i\sqrt{\varepsilon})})^{-1}$ and $\omega = e^{\pi i/4}\sqrt{1 + i\sqrt{\varepsilon}}$. Thus, the boundary layer thickness is $O(\sqrt[4]{\varepsilon})$. Finally, if $E_{10}B_{10}$ is singular, but nonzero (as when $n < m_1$), progress could be generally made through preliminary algebraic manipulations (cf. Anderson (1973) for an analogous control problem).

3. Construction of Asymptotic Solutions.

A linear nonhomogeneous boundary value problem can be solved by variation of parameters once a complete set of linearly independent solutions of the corresponding homogeneous system is known. For the asymptotic solution of (1), we'd need $n + m$ linearly independent asymptotic solutions. Alternatively, one could seek an outer solution of (1) (i.e., a regular perturbation of an already obtained solution of the reduced system (7)) and modify it by adding the appropriate boundary layer corrections (cf. O'Malley (1969b) which solves corresponding scalar problems). Since we can supply $n - m_1$ boundary conditions (like (16)) for the reduced system, we can expect an outer solution under hypotheses (H1) - (H3) to be of the form

$$(17)\qquad (X(t,\sqrt{\varepsilon}),Y_1(t,\sqrt{\varepsilon}),Y_2(t,\sqrt{\varepsilon}),Y_3(t,\sqrt{\varepsilon}))$$

$$\sim \sum_{j=0}^{\infty} (X_j(t),Y_{1j}(t),Y_{2j}(t),Y_{3j}(t))\varepsilon^{j/2}$$

being an asymptotic solution within $(0,1)$ which converges to a solution $(X_0,Y_{10},Y_{20},Y_{30})$ of the reduced system (7) as $\varepsilon \to 0$. It would be completely determined termwise by the boundary values

$$(18)\qquad E(j)X(j,\sqrt{\varepsilon}), \quad j = 0 \text{ or } 1$$

since higher order terms will satisfy a system of the form (7) with successively known nonhomogeneous terms.

Since E has rank $n - m_1$, the outer solution is parameterized by $n - m_1$ vector functions of $\sqrt{\varepsilon}$, and there is need for $m + n - (n - m_1) = m + m_1$ linearly independent boundary layer solutions of the homogeneous

version of (6) which are asymptotically negligible within $(0,1)$. For the regular problem with $m_2 + m_3 = m$, there would be m_3 boundary layer solutions which are decaying functions of the stretched variable $\kappa = t/\varepsilon$ and m_2 which are decaying functions of $\rho = (1-t)/\varepsilon$ (cf., say, Harris (1973)). For our singular problems, however, we shall also find m_1 boundary layer solutions depending on each of the stretched variables $\tau = t/\sqrt{\varepsilon}$ and $\sigma = (1-t)/\sqrt{\varepsilon}$. (These thicker ($\sqrt{\varepsilon} >> \varepsilon$) boundary layers (upon matching) now require our asymptotic expansions to all be power series in

$$\mu = \sqrt{\varepsilon}, \tag{19}$$

rather than ε. In order to generate these asymptotic solutions, we'll assume the coefficients in (1) to be infinitely differentiable, though finite approximations could be obtained under less smoothness.

We shall construct asymptotic solutions to (6) of the form

$$\left\{\begin{aligned} x(t,\varepsilon) &= X(t,\mu) + \mu^{\alpha} m(\tau,\mu) + \mu^{\beta+1} n(\sigma,\mu) + \mu^{\gamma+2} r(\rho,\mu) + \mu^{\delta+2} j(\kappa,\mu) \\ y_1(t,\varepsilon) &= Y_1(t,\mu) + \mu^{\alpha-1} p_1(\tau,\mu) + \mu^{\beta} q_1(\sigma,\mu) + \mu^{\gamma+2} s_1(\rho,\mu) + \mu^{\delta+2} \ell_1(\kappa,\mu) \\ y_2(t,\varepsilon) &= Y_2(t,\mu) + \mu^{\alpha} p_2(\tau,\mu) + \mu^{\beta+1} q_2(\sigma,\mu) + \mu^{\gamma} s_2(\rho,\mu) + \mu^{\delta+2} \ell_2(\kappa,\mu) \\ y_3(t,\varepsilon) &= Y_3(t,\mu) + \mu^{\alpha} p_3(\tau,\mu) + \mu^{\beta+1} q_3(\sigma,\mu) + \mu^{\gamma+2} s_3(\rho,\mu) + \mu^{\delta} \ell_3(\kappa,\mu) \end{aligned}\right. \tag{20}$$

where the functions of the stretched variables

$$\tau = t/\mu, \quad \sigma = (1-t)/\mu, \quad \rho = (1-t)/\mu^2, \quad \text{and} \quad \kappa = t/\mu^2 \tag{21}$$

tend to zero as the appropriate variable tends to infinity. The outer solu-

tion (17), then, provides the asymptotic solution within $(0,1)$. The scalings μ^{α}, μ^{β}, μ^{γ}, and μ^{δ} for the boundary layer corrections remain free to meet various boundary conditions (2), while the remaining μ^{-1}, μ, and μ^2 factors are simply used to prevent calculation of trivial coefficients.

Since the outer solution (17) satisfies the nonhomogeneous system (6), the boundary layer corrections will satisfy the corresponding homogeneous system. The boundary layer solutions which depend on κ or ρ are completely analogous to those obtained for the regular problem when $m_1 = 0$.

Let us first seek m_1 boundary layer solutions of the homogeneous system (6) of the form

$$(22)\qquad (m, \frac{1}{\mu} p_1, p_2, p_3) \sim \sum_{k=0}^{\infty} (m_k(\tau), \frac{1}{\mu} p_{1k}(\tau), p_{2k}(\tau), p_{3k}(\tau))\mu^k$$

(cf. (20)). Thus, we'll have

$$(23)\qquad \begin{cases} \dfrac{1}{\mu}\dfrac{dm}{d\tau} = Am + \dfrac{1}{\mu} B_1 p_1 + B_2 p_2 + B_3 p_3 \\ \dfrac{dp_1}{d\tau} = E_1 m + \mu F_1 p_1 + \mu^2 F_2 p_2 + \mu^2 F_3 p_3 \\ \mu \dfrac{dp_2}{d\tau} = E_2 m + \mu F_4 p_1 + F_5 p_2 + \mu^2 F_6 p_3 \\ \mu \dfrac{dp_3}{d\tau} = E_3 m + \mu F_7 p_1 + \mu^2 F_8 p_2 + F_9 p_3. \end{cases}$$

When $\mu = 0$, this reduces to the limiting problem

$$\begin{cases} \dfrac{dm_0}{d\tau} = B_{10}(0) p_{10} \\ \dfrac{dp_{10}}{d\tau} = E_{10}(0) m_0 \end{cases}$$

$$(24)\qquad \begin{cases} 0 = E_{20}(0)m_0 + F_{50}(0)p_{20} \\ 0 = E_{30}(0)m_0 + F_{90}(0)p_{30} \end{cases}$$

so

$$(25)\qquad \begin{cases} p_{20}(\tau) = -F_{50}^{-1}(0)E_{20}(0)m_0(\tau) \\ p_{30}(\tau) = -F_{90}^{-1}(0)E_{30}(0)m_0(\tau) \end{cases}$$

while

$$(26)\qquad \frac{d^2 m_0}{d\tau^2} = B_{10}(0)E_{10}(0)m_0.$$

Since $\mathcal{E}(0)B_{10}(0) = 0$, $\frac{d^2}{d\tau^2}(\mathcal{E}(0)m_0) = 0$, so the only solution m_0 which decays to zero together with its first derivative as $\tau \to \infty$ must satisfy

$$(27)\qquad \mathcal{E}(0)m_0(\tau) = 0.$$

Furthermore, multiplying (26) by $E_{10}(0)$, provides the decaying solution

$$(28)\qquad E_{10}(0)m_0(\tau) = e^{-\sqrt{E_{10}(0)B_{10}(0)}\,\tau}\, E_{10}(0)m_0(0),$$

and the definition (12) of $\mathcal{E}$ implies that

$$(29)\qquad m_0(\tau) = Q(0)e^{-\sqrt{E_{10}(0)B_{10}(0)}\,\tau}\, E_{10}(0)m_0(0).$$

Finally, the differential equation for p_{10} has the decaying solution

(30) $$p_{10}(\tau) = -(\sqrt{E_{10}(0)B_{10}(0)})^{-1}E_{10}(0)m_0(\tau).$$

Since $p_{10}(0)$ and $E_{10}(0)m_0(0)$ are arbitrary, we are able to provide m_1 linearly independent solutions to (24) by specifying either. Higher order coefficients in (22) satisfy nonhomogeneous forms of (24) with successively known, exponentially decaying terms. The decaying solutions (22) are thereby completely determined up to specification of either

(31) $$p_1(0,\mu) \quad \text{or} \quad E_{10}(0)m(0,\mu).$$

This possibility of a choice of boundary values makes the boundary layer corrections with $O(\mu)$ boundary layers more flexible than those with $O(\varepsilon)$ layers (cf. (40)).

The classical boundary layer correction

(32) $$(\mu^2 j, \mu^2 \ell_1, \mu^2 \ell_2, \ell_3) \sim \sum_{m=0}^{\infty} (\mu^2 j_m, \mu^2 \ell_{1m}, \mu^2 \ell_{2m}, \ell_{3m})\mu^m$$

must satisfy the system

(33) $$\begin{cases} \dfrac{dj}{d\kappa} = \mu^2 Aj + \mu^2 B_1 \ell_1 + \mu^2 B_2 \ell_2 + B_3 \ell_3 \\[2mm] \dfrac{d\ell_1}{d\kappa} = E_1 j + \mu^2 F_1 \ell_1 + \mu^2 F_2 \ell_2 + F_3 \ell_3 \\[2mm] \dfrac{d\ell_2}{d\kappa} = E_2 j + \mu^2 F_4 \ell_1 + F_5 \ell_2 + F_6 \ell_3 \\[2mm] \dfrac{d\ell_3}{d\kappa} = \mu^2 E_3 j + \mu^4 F_7 \ell_1 + \mu^4 F_8 \ell_2 + F_9 \ell_3. \end{cases}$$

The resulting limiting problem

$$(34)\quad \begin{cases} \dfrac{dj_0}{d\kappa} = B_{30}(0)\ell_{30}, & \dfrac{d\ell_{10}}{d\kappa} = E_{10}(0)j_0 + F_{30}(0)\ell_{30} \\ \dfrac{d\ell_{20}}{d\kappa} = E_{20}(0)j_0 + F_{50}(0)\ell_{20} + F_{60}(0)\ell_{30}, & \dfrac{d\ell_{30}}{d\kappa} = F_{90}(0)\ell_{30} \end{cases}$$

has the unique decaying solution

$$(35)\quad \begin{cases} \ell_{30}(\kappa) = e^{F_{90}(0)\kappa}\ell_{30}(0), \\ j_0(\kappa) = B_{30}(0)F_{90}^{-1}(0)\ell_{30}(\kappa), \\ \ell_{10}(\kappa) = (E_{10}(0)B_{30}(0)F_{90}^{-2}(0) + F_{30}(0)F_{90}^{-1}(0))\ell_{30}(\kappa), \\ \text{and} \\ \ell_{20}(\kappa) = -\int_{\kappa}^{\infty} e^{F_{50}(0)(\kappa-s)}(E_{20}(0)B_{30}(0)F_{90}^{-1}(0) + F_{60}(0))\ell_{30}(s)ds \end{cases}$$

up to selection of $\ell_{30}(0)$. Higher order terms satisfy nonhomogeneous forms of (34) and are completely determined up to the initial value

$$(36)\qquad \ell_3(0,\mu).$$

This procedure, then, allows m_3 linearly independent boundary layer solutions (32) to be constructed.

The terminal boundary layer solutions are found in a manner analogous to those at $t = 0$. Thus, the leading terms of

$$(37)\qquad (\mu n, q_1, \mu q_2, \mu q_3) \sim \sum_{m=0}^{\infty} (\mu n_m, q_{1m}, \mu q_{2m}, \mu q_{3m})\mu^m$$

are

$$(38)\quad \begin{cases} q_{20}(\sigma) = -F_{50}^{-1}(1)E_{20}(1)n_0(\sigma), \\ q_{30}(\sigma) = -F_{90}^{-1}(1)E_{30}(1)n_0(\sigma), \\ q_{10}(\sigma) = e^{-\sqrt{E_{10}(1)B_{10}(1)}\sigma} q_{10}(0) \\ n_0(\sigma) = B_{10}(1)(\sqrt{E_{10}(1)B_{10}(1)})^{-1} q_{10}(\sigma). \end{cases}$$

We note that (12) implies that

$$(39)\qquad E(1)n_0(\sigma) = 0$$

so that $n_0(\sigma) = Q(1)E_{10}(1)n_0(\sigma)$ and these leading terms are therefore determined up to the initial value $q_{10}(0) = (\sqrt{E_{10}(1)B_{10}(1)})^{-1}E_{10}(1)n_0(0)$. Analogous work successively provides the higher order terms in (37) up to specification of either

$$(40)\qquad q_1(0,\mu) \quad \text{or} \quad E_{10}(1)n(0,\mu).$$

Finally, the leading terms of the boundary layer correction

$$(41)\qquad (\mu^2 r, \mu^2 s_1, s_2, \mu^2 s_3) \sim \sum_{m=0}^{\infty} (\mu^2 r_m, \mu^2 s_{1m}, s_{2m}, \mu^2 s_{3m})\mu^m$$

are given by

$$\begin{cases} s_{20}(\rho) = e^{-F_{50}(1)\rho} s_{20}(0) \\ r_0(\rho) = B_{20}(1)F_{50}^{-1}(1)s_{20}(\rho) \end{cases}$$

$$
(42) \quad \begin{cases} s_{10}(\rho) = (E_{10}(1)B_{20}(1)F_{50}^{-2}(1) + F_{20}(1)F_{50}^{-1}(1))s_{20}(\rho) \\ s_{30}(\rho) = \int_{\rho}^{\infty} e^{F_{90}(1)(t-\rho)}(E_{30}(1)B_{20}(1)F_{50}^{-1}(1) + F_{80}(1))s_{20}(t)dt \end{cases}
$$

and the solution is completely specified up to the initial m_2 vector

$$
(43) \qquad s_2(0,\mu).
$$

We could prove the asymptotic validity of the solutions (20) which we've constructed by integral equations methods (cf. Harris (1973) or Vasil'eva and Butuzov (1973)). Although that proof would differ somewhat from the classical (regular) ones, we regard the construction of the solutions as the most challenging aspect of this study and shall not further discuss the details of proof.

4. Fitting the Boundary Conditions.

Since our construction of the outer solution (17) and of the thicker boundary layer corrections (22) and (37) distinguish between components of Ex and $E_{10}x$, it is natural to write

$$
(44) \qquad x = x_1 + Qx_2
$$

for the n and m_1 dimensional vectors

$$
(45) \qquad x_1 = E(t)x(t,\varepsilon) \quad \text{and} \quad x_2 = E_{10}(t)x(t,\varepsilon).
$$

(We experienced a similar separation of components in solving the reduced

problem (7) where we had a differential equation for $X_{10} = EX_0$ and a linear algebraic equation for $X_{20} = E_{10}X_0$.) Instead of the x representation of (20), then, it will be more convenient to use the further decomposition

$$(46)\quad \begin{cases} x_1(t,\varepsilon) = X_1(t,\mu) + \mu^{\alpha+1}m_1(\tau,\mu) + \mu^{\beta+2}n_1(\sigma,\mu) + \mu^{\gamma+2}r_1(\rho,\mu) + \mu^{\delta+2}j_1(\kappa,\mu) \\ x_2(t,\varepsilon) = X_2(t,\mu) + \mu^{\alpha}m_2(\tau,\mu) + \mu^{\beta+1}n_2(\sigma,\mu) + \mu^{\gamma+2}r_2(\rho,\mu) + \mu^{\delta+2}j_2(\kappa,\mu) \end{cases}$$

where, e.g., $X_1(t,\mu) = E(t)X(t,\mu)$ and $m_1(\tau,\mu) = \frac{1}{\mu} E(\mu\tau)m(\tau,\mu) = O(1)$ by (27). We note that $E_{10}E = 0$ implies that we must have

$$(47)\qquad E_{10}x_1 = 0.$$

Now note that the solution $(x'\ y')'$ of the original system (1) is of the form

$$(48)\qquad \begin{pmatrix} x \\ y \end{pmatrix} = T(t)u(t)$$

where

$$(49)\qquad T(t) = \begin{pmatrix} I_n & Q & 0 & 0 & 0 \\ 0 & 0 & P_1 & P_2 & P_3 \end{pmatrix}$$

for

$$P = (P_1\ P_2\ P_3)$$

and u is the $n + m_1 + m$ dimensional vector

$$u(t) = (x_1' \; x_2' \; y_1' \; y_2' \; y_3')'. \tag{50}$$

Furthermore, the expansions (42) imply that all solutions (20) of the transformed problem (6) are given by

$$u(t) = U(t,\mu)k(\mu) \tag{51}$$

where the square matrix U is

$$U(t,\mu) = \begin{pmatrix} X_1(t,\mu) & \mu m_1(\tau,\mu) & \mu^2 n_1(\sigma,\mu) & \mu^2 r_1(\rho,\mu) & \mu^2 j_1(\kappa,\mu) \\ X_2(t,\mu) & m_2(\tau,\mu) & \mu n_2(\sigma,\mu) & \mu^2 r_2(\rho,\mu) & \mu^2 j_2(\kappa,\mu) \\ Y_1(t,\mu) & \frac{1}{\mu} p_1(\tau,\mu) & q_1(\sigma,\mu) & \mu^2 s_1(\rho,\mu) & \mu^2 \ell_1(\kappa,\mu) \\ Y_2(t,\mu) & p_2(\tau,\mu) & \mu q_2(\sigma,\mu) & s_2(\rho,\mu) & \mu^2 \ell_2(\kappa,\mu) \\ Y_3(t,\mu) & p_3(\tau,\mu) & \mu q_3(\sigma,\mu) & s_3(\rho,\mu) & \ell_3(\kappa,\mu) \end{pmatrix} \tag{52}$$

with fixed boundary values

$$\begin{cases} X_1(j,\mu) = I_n, \quad j = 0 \text{ or } 1, \quad m_2(0,\mu) = I_{m_1} = q_1(0,\mu) \\ s_2(0,\mu) = I_{m_2}, \quad \text{and} \quad \ell_3(0,\mu) = I_{m_3}, \end{cases} \tag{53}$$

and the $n + m + m_1$ vector $k(\mu)$ is partitioned as

(54) $$k(\mu) = (k_1'(\mu)\ \mu^{\alpha}k_2'(\mu)\ \mu^{\beta}k_3'(\mu)\ \mu^{\gamma}k_4'(\mu)\ \mu^{\delta}k_5'(\mu))'.$$

We note that the boundary values (53) imply that $p_{10}(0) = -(\sqrt{E_{10}(0)B_{10}(0)})^{-1}$, and $n_{20}(0) = (\sqrt{E_{10}(1)B_{10}(1)})^{-1}$, so both $p_1(0,\mu)$ and $n_2(0,\mu)$ are nonsingular for ε sufficiently small. Also note that $U(t,\mu)$ would be a fundamental matrix if the corresponding $n + m + m_1$ dimensional system were homogeneous. We shall select α,β,γ, and δ so that the $k_i(0)$'s are $O(1)$ as $\varepsilon \to 0$ and nonzero if the particular k_i is not identically zero.

Putting (48) and (51) together, the boundary conditions (2) imply the $n + m$ linear equations

(55) $$\Delta_1(\mu)k(\mu) = c(\varepsilon)$$

for the unknowns $k_i(\mu)$ where

(56) $$\Delta_1(\mu) = \sum_{j=0}^{1} (R_j(\mu^2)\ S_j(\mu^2))T(j)U(j,\mu).$$

An additional m_1 boundary conditions result if we impose the endcondition

(57) $$\Delta_2(\mu)k(\mu) \equiv E_{10}(j)x_1(j,\mu^2) = 0, \qquad j = 0 \text{ or } 1,$$

required by consistency with (47). (Note that E_{10} has rank m_1.) Thus, the boundary value problem (1) - (2) will have a unique asymptotic solution of the form

(58) $$\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \end{pmatrix} = T(t)U(t,\mu)\Delta^{-1}(\mu)\begin{pmatrix} c(\mu^2) \\ 0 \end{pmatrix}$$

provided

(H4) <u>the</u> $(n+m+m_1)\times(n+m+m_1)$ <u>matrix</u> $\Delta(\mu) = (\Delta_1'(\mu)\ \Delta_2'(\mu))'$ <u>is nonsingular for</u> ε <u>sufficiently small</u>.

Because the boundary layer correction terms are asymptotically negligible away from one endpoint, we can considerably simplify calculation of $\Delta(\mu)$. Thus, up to asymptotically exponentially small terms,

$T(0)U(0,\mu) =$

$$\begin{pmatrix} X_1(0,\mu) + Q(0)X_2(0,\mu) & \mu m_1(0,\mu) + Q(0) & 0 & 0 & \mu^2 j_1(0,\mu) + Q(0)j_2(0,\mu) \\ \sum_{\ell=1}^{3} P_\ell(0)Y_\ell(0,\mu) & \frac{1}{\mu}P_1(0)p_1(0,\mu) + \sum_{\ell=2}^{3} P_\ell(0)p_\ell(0,\mu) & 0 & 0 & \sum_{k=1}^{2}\mu^2 P_k(0)\ell_k(0,\mu) + P_3(0) \end{pmatrix}$$

and likewise for $T(1)U(1,\mu)$. These imply that

(59) $$\Delta(\mu) = (\Delta_{k\ell}),\quad k = 1, 2;\quad \ell = 1, \dots, 5$$

where

$$\Delta_{11} \sim \sum_{j=0}^{1} \{R_j(X_1(j,\mu) + Q(j)X_2(j,\mu)) + S_j \sum_{\ell=1}^{3} P_\ell(j)Y_\ell(j,\mu)\},$$

$$\Delta_{12} \sim R_0(\mu m_1(0,\mu) + Q(0)) + \frac{1}{\mu} S_0(P_1(0)p_1(0,\mu) + \mu \sum_{\ell=2}^{3} P_\ell(0)p_\ell(0,\mu)),$$

$$\Delta_{13} \sim R_1(\mu^2 n_1(0,\mu) + \mu Q(1)n_2(0,\mu)) + S_1(P_1(1)q_1(0,\mu) + \mu \sum_{\ell=2}^{3} P_\ell(1)q_\ell(0,\mu)),$$

$$\Delta_{14} \sim \mu^2 R_1(r_1(0,\mu) + Q(1)r_2(0,\mu)) + S_1(\mu^2 P_1(1)s_1(0,\mu) + \sum_{\ell=2}^{3} P_\ell(1)s_\ell(0,\mu)),$$

$$\Delta_{15} \sim R_0(\mu^2 j_1(0,\mu) + Q(0)j_2(0,\mu)) + S_0(\mu^2 \sum_{k=1}^{2} P_k(0)\ell_k(0,\mu) + P_3(0)),$$

$\Delta_{21} \sim E_{10}(j)X_1(j,\mu)$ with j determined in (57),

and

$$(\Delta_{22}, \Delta_{23}, \Delta_{24}, \Delta_{25}) \sim \begin{cases} \mu E_{10}(0)\ (m_1(0,\mu),\ 0,\ 0,\ \mu j_1(0,\mu)) & \text{if } j = 0 \\ \text{and} & \\ \mu^2 E_{10}(1)\ (0,\ n_1(0,\mu),\ r_1(0,\mu),\ 0) & \text{if } j = 1. \end{cases}$$

Because $\Delta_{12} = 0(1/\mu)$ $(= 0(1)$ only if $S_1P_1(0)p_{10}(0) = 0)$, the matrix $\Delta(\mu)$ will have an asymptotic series expansion

$$\Delta(\mu) \sim \frac{1}{\mu} \sum_{j=0}^{\infty} \delta_j \mu^j. \tag{60}$$

It will therefore be nonsingular if the limiting matrix δ_0 is nonsingular, although $\Delta(\mu)$ can still be nonsingular for $\varepsilon \neq 0$ if $\delta_0 = 0$. If δ_ℓ is the first nonsingular coefficient in (60), $\Delta^{-1}(\mu)$ will be $0(\mu^{1-\ell})$, so the solution (58) of the given problem will generally be unbounded like $0(\mu^{-\ell})$. In particular, note that a bounded solution will result if $\ell = 0$ and that the powers α,β,γ, and δ in (54) are integers. Further, the limiting solution within $(0,1)$ depends only on $k_1(\mu)$, so we might say which boundary conditions are appropriate for the reduced problem (7) (cf. Harris (1973)).

We might also consider the possibility of nonunique solutions under appropriate orthogonality assumptions if $\Delta(\mu)$ is singular.

To summarize our principal results, we have

Theorem:

Under hypotheses (H1) - (H4), *we obtain a unique solution* (58) *of the boundary value problem* (1) - (2).

5. Natural Boundary Value Problems.

a) Sample Problem 1.

Suppose we are given a problem in the transformed form (6) with prescribed boundary values

$$(61) \qquad x(0), \quad y_1(1), \quad y_2(1), \quad \text{and} \quad y_3(1).$$

Instead of actually obtaining $\Delta^{-1}(\mu)$, we can apply the boundary conditions in (51) to obtain a solution with

$$(62) \qquad \begin{cases} k_1(0) = E(0)x(0) \quad \text{for} \quad X_1(0,\mu) = I_n, \\ \alpha = 0, \quad k_2(0) = E_{10}(0)x(0) - X_{20}(0)k_1(0), \\ \beta = 0, \quad k_3(0) = y_1(1) - Y_{10}(1)k_1(0), \\ \gamma = 0, \quad k_4(0) = y_2(1) - Y_{20}(1)k_1(0), \\ \delta = 0, \quad k_5(0) = y_3(0) - Y_{30}(0)k_1(0) - P_{30}(0)k_2(0). \end{cases}$$

In particular, the limiting solution within $(0,1)$ will satisfy the reduced

problem (7) and the initial condition (16) with $j = 0$.

b) Sample Problem 2.

Suppose our problem is of the transformed form (6) with prescribed boundary values

$$(63)\qquad x(1), \quad y_1(0), \quad y_2(1), \quad \text{and} \quad y_3(0).$$

Again, the unique solution is readily found to be of the form (51) with

$$(64)\quad \left\{\begin{array}{ll} & k_1(0) = E(1)x(1) \quad \text{for} \quad X_1(1,\mu) = I_n, \\ \alpha = 1, & k_2(0) = -\sqrt{E_{10}(0)B_{10}(0)}\,(y_1(0) - Y_{10}(0)k_1(0)), \\ \beta = -1, & k_3(0) = \sqrt{E_{10}(1)B_{10}(1)}\,(E_{10}(1)x(1) - X_{20}(1)k_1(0)), \\ \gamma = 0, & k_4(0) = y_2(1) - Y_{20}(1)k_1(0) - g_{20}(0)k_3(0), \\ \delta = 0, & k_5(0) = y_3(0) - Y_{30}(0)k_1(0) \end{array}\right.$$

and the limiting solution within $(0,1)$ satisfies the system (7) and the terminal condition (16) with $j = 1$.

c) The General Problem.

As our special problems suggest, we can seek a solution (51) of the transformed system (6) plus boundary conditions. That problem will be uniquely solvable provided the reduced system (7) can be uniquely solved subject to appropriate boundary conditions. We must suppose that $E_{10}(0)x(0,\varepsilon)$ or $y_1(0,\varepsilon)$, $E_{10}(1)x(1,\varepsilon)$ or $y_1(1,\varepsilon)$, $y_2(1,\varepsilon)$, and $y_3(1,\varepsilon)$ can be obtained in order to uniquely determine the initial values (31), (36), (40), and (43) for the various boundary layer corrections of the

solution (20) - (46). In order to solve the reduced problem (7), separated boundary values for $E(j)X(j,\varepsilon)$ need not be given (as in the sample problems). However, if the prescribed boundary values for Ex are coupled at $t = 0$ and $t = 1$, existence of the solution to the resulting reduced two point problem is not a priori guaranteed. One must always be able to solve the given conditions for $y_2(1,\varepsilon)$ and $y_3(0,\varepsilon)$. Knowing $y_3(0,\varepsilon)$, for example, one would solve

$$y_3(0,\varepsilon) \sim Y_3(0,\mu)k_1(\mu) + \mu^{\alpha}p_3(0,\mu)k_2(\mu) + \mu^{\delta}k_5(\mu)$$

to obtain k_5.

Further, it is essential that at least $m_1 + m_3$ boundary values be obtainable at $t = 0$ and at least $m_1 + m_2$ boundary values be obtainable at $t = 1$ because these are the number of linearly independent boundary layer corrections decaying at those endpoints. In particular, we cannot expect to asymptotically solve initial value problems or terminal value problems for (1) unless we artificially restrict boundary values to appropriate lower dimensional manifolds (cf. Hoppensteadt (1971)).

Since the solution of general problems (1) - (2) relates crucially to the solution of simpler transformed problems (like our sample problems), it is convenient to solve general problems in terms of simpler "natural" ones. This generalized "shooting" method has been somewhat developed by Keller and White (1975) and Ferguson (1975).

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~~We consider~~ the asymptotic solution of boundary value problems are considered for the vector system

dx/dt →
$$\dot{x} = A(t,\varepsilon)x + B(t,\varepsilon)y + C(t,\varepsilon)$$
epsilon

epsilon dy/dt →
$$\varepsilon\dot{y} = E(t,\varepsilon)x + F(t,\varepsilon)y + G(t,\varepsilon)$$

epsilon approaches 0

as $\varepsilon \to 0$ under the assumption that the matrix $F(t,0)$ is singular. A full set of asymptotic solutions is constructed assuming that $F(t,0)$ can → next page

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be block-diagonalized, the reduced problem is consistent, and a new stability condition holds. Boundary value problems are then solvable if an appropriate "boundary" matrix is nonsingular for $\varepsilon \neq 0$. Such problems arise in optimal control theory, among other applications.

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